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ASYMPTOTIC ANALYSIS OF INVISCID PERTURBATIONS IN A SUPERSONIC BOUNDARY LAYER

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UDC 532.526

Inviscid perturbations play an important role in the linear theory of stability of supersonic boundary layers. These perturbations are described by nonsteady linearized Euler equations [1, 2]. In numerical calculations conducted in [3] for inviscid two-dimensional perturbations in a plane-parallel boundary layer on a thermally insulated plate, it was found that for incoming flows with a Mach number $M \geqslant 3$, other modes besides the first unstable mode are manifest. The number of modes increases rapidly with an increase in M, the short-wave (high-frequency) part of the spectrum being filled here. It was found in experiments [4] that the second mode begins to dominate the first mode in the boundary layer on a cone for $M \gtrsim 5.6$. The author of [5] recorded unstable high-frequency perturbations in supersonic flow about a cone. In [1], it was suggested that these disturbances are associated with the third and higher modes. The question of the role of higher modes in the agitation of a supersonic boundary layer has not yet been resolved. In connection with this, it is interesting to study their properties both theoretically and experimentally.

Here, we perform an asymptotic analysis of inviscid perturbations in a shortwave approximation. We find the dispersion relation and eigenfunctions for neutral modes with large numbers. Numerical calculations are performed to obtain the stability characteristics for the first four modes in a plane-parallel boundary layer with M = 8. It is shown that the

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 69-75, January-February, 1989. Original article submitted November 10, 1987.

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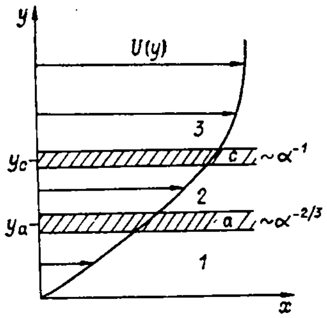

Fig. 1

neutral solutions obtained by direct calculation agree well with the asymptotic theory beginning with the mode number n ≥ 3. It is concluded on the basis of the numerical and asymptotic analyses that an increase in n is accompanied by a rapid decrease in the growth increments and that, in the limit, inviscid perturbations become neutral free oscillations of an acoustic nature.

1. We will examine laminar flow in a supersonic plane-parallel boundary layer of a perfect gas. The y axis is directed along a normal to the surface in the flow, the x axis is directed downflow, and the z axis coincides with the transverse direction. The coordinates x, y, and z are referred to the characteristic length L, which is commensurate with the thickness of the boundary layer. It is assumed that, in the undisturbed flow, the normal and transverse components of velocity are equal to zero, while pressure is constant. The profiles of the x-components of velocity U(y) and temperature T(y) are made dimensionless with respect to the velocity U_∞ and temperature T_∞ of the incoming flow.

A small nonsteady perturbation of the form $\mathbf{G}(y)\exp(i\alpha x + i\beta z - i\omega t)$ is superimposed on the main flow, where α and β are the wave numbers in the x- and z-directions; ω is the frequency, referred to U_∞/L; $\mathbf{G} = (u, v, w, p, \theta)$ is a vector function whose components describe the amplitude of the perturbation of the x-, y-, and z-components of velocity, pressure, and temperature, respectively. To find the free vibrations, we impose the condition of decay at $y \to \infty$ and impermeability at y = 0 on the perturbation. In the inviscid limit, the linearized equations for the amplitude $\mathbf{G}$ can be reduced to a second-order equation for the pressure perturbation p(y). This equation, together with the boundary conditions, constitutes the boundary-value problem [2]

$$p'' - \left(\frac{2U'}{U-c} - \frac{T'}{T}\right)p' + \alpha^2\left[\frac{M^2(U-c)^2}{T} - \varkappa^2\right]p = 0,\ p'(0) = 0,\qquad (1.1)$$
$$p(y) \to 0,\quad y \to \infty.$$

Here $\varkappa^2 = 1 + \beta^2/\alpha^2$; $c = \omega/\alpha$ is the phase velocity. The imaginary part of the phase velocity c_i determines the increments of growth ($c_i > 0$) or decay ($c_i < 0$) of free oscillations of the boundary layer.

We will study neutral subsonic perturbations for which c is a real quantity satisfying the condition $1 - \varkappa/M < c < 1 + \varkappa/M$. Using the method of combinable asymptotic expansions, we analyze shortwave perturbations corresponding to the limit $\alpha \to \infty$, M > 1 is fixed. The structure of the solutions of boundary-value problem (1.1) is determined by the position of the singular point y_c ($U(y_c) = c$) and the turning points y_a, which satisfy the equation

$$M^2[U(y_a) - c]^2/T(y_a) - \varkappa^2 = 0.\qquad (1.2)$$

The singular point y_c is the coordinate of the critical layer. The physical significance of the turning points y_a is best understood by using the example of a plane wave ($\varkappa = 1$). Then (1.2) can be written in the form $U(y_a) = c \pm a(y_a)$ (a is the local speed of sound, referred to U_∞). It is evident that, at the turning points, the velocity of the flow relative to an observer moving with the phase velocity of the perturbation is equal to the speed of sound. Thus, y_a are often called acoustic points [2].

Figure 1 shows the structure of characteristic regions of change of the solution for subsonic perturbations with the phase velocity $1 - \varkappa/M < c < 1$. In the neighborhood of the turning point y_a ($U(y_a) = c - \varkappa T^{1/2}(y_a)/M$) , we distinguish a layer with the characteristic scale $\delta_a \sim \alpha^{-2/3}$. A layer with the scale $\delta_c \sim \alpha^{-1}$ is formed near the critical point. The layers are separated by regions 1, 2, and 3 with the scale $\sim O(1)$.

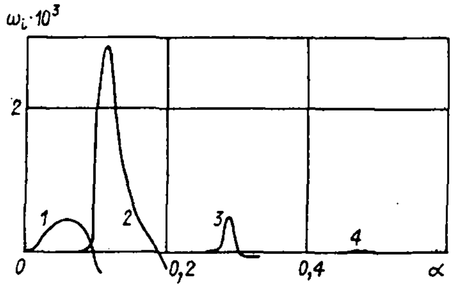

Fig. 2

Let us perform the transformation $p(y) = T^{-1/2}(U - c)u(y)$. Then system (1.1) takes the form

$$u'' + \alpha^2[q(y) + \alpha^{-2}q_1(y)]u = 0,\ u'(0) = 0,\ u(y) \to 0,\ y \to \infty, \tag{1.3}$$
$$q(y) = M^2(U - c)^2/T - \varkappa^2,\ q_1(y) = T'^2/4T^2 - 2U'^2/(U - c)^2 + (U'T)'/T(U - c) - T''/2T.$$

In regions 1, 2, and 3, WKB-expansions [6] are valid:

$(0 < y < y_a,\ q > 0)$

$$u_1(y) = q^{-1/4}\left[C_1\cos\left(\alpha\int_y^{y_a}\sqrt{q}\,dy\right) + C_2\sin\left(\alpha\int_y^{y_a}\sqrt{q}\,dy\right)\right](1+O(\alpha^{-1})); \tag{1.4}$$

$(y_a < y < y_c,\ q < 0)$

$$u_2(y) = (-q)^{-1/4}\left[A_1\exp\left(\alpha\int_{y_a}^{y}\sqrt{-q}\,dy\right) + A_2\exp\left(-\alpha\int_{y_a}^{y}\sqrt{-q}\,dy\right)\right](1 + O(\alpha^{-1})); \tag{1.5}$$

$(y_c < y < \infty,\ q < 0)$

$$u_3(y) = (-q)^{-1/4}\left[E_1\exp\left(\alpha\int_{y_c}^{y}\sqrt{-q}\,dy\right) + E_2\exp\left(-\alpha\int_{y_c}^{y}\sqrt{-q}\,dy\right)\right](1 + O(\alpha^{-1})). \tag{1.6}$$

In the neighborhood of a turning point, we change over to the internal variable $\xi = (-q_a')^{1/3}(y - y_a)\alpha^{2/3}$ $(q_a' = q'(y_a) < 0)$ Then $d^2u/d\xi^2 - [\xi + O(\alpha^{-2/3})]u = 0$.

In the principal approximation, the solution is expressed through the Airy function [7]

$$u(\xi) = B_1Ai(\xi) + B_2Bi(\xi). \tag{1.7}$$

The following asymptotic solutions are valid on the boundaries of the transitional layer $(\xi \to \pm\infty)$

$$Ai(\xi) \sim \frac{\exp\left(-\frac{2}{3}\xi^{3/2}\right)}{2\pi^{1/2}\xi^{1/4}},\quad Bi(\xi) \sim \frac{\exp\left(\frac{2}{3}\xi^{3/2}\right)}{\pi^{1/2}\xi^{1/4}},\quad \xi \to +\infty,$$
$$Ai(\xi) \sim \frac{\sin\left[\frac{2}{3}(-\xi)^{3/2} + \frac{\pi}{4}\right]}{\pi^{1/2}(-\xi)^{1/4}},\quad Bi(\xi) \sim \frac{\cos\left[\frac{2}{3}(-\xi)^{3/2} + \frac{\pi}{4}\right]}{\pi^{1/2}(-\xi)^{1/4}},\quad \xi \to -\infty.$$

In the critical layer with the internal variable $\eta = \varkappa\alpha(y - y_c)$, we have $\frac{d^2p}{d\eta^2} - \left[\frac{2}{\eta} + O(\alpha^{-1})\right] \times \frac{dp}{d\eta} - [1 + O(\alpha^{-2})]p = 0.$ In the principal approximation

$$p = D_1(\eta - 1)e^{\eta} + D_2(\eta + 1)e^{-\eta}. \tag{1.8}$$

If we combine solutions (1.4), (1.7), and (1.5) between zones 1, a, and 2, we find the relationship between the coefficients C_j, A_j, B_j: $C_{1,2} = (-q_a')^{1/6}\alpha^{-1/6}\pi^{-1/2}\frac{\sqrt{2}}{2}(B_1 \pm B_2)$, $A_1 =$

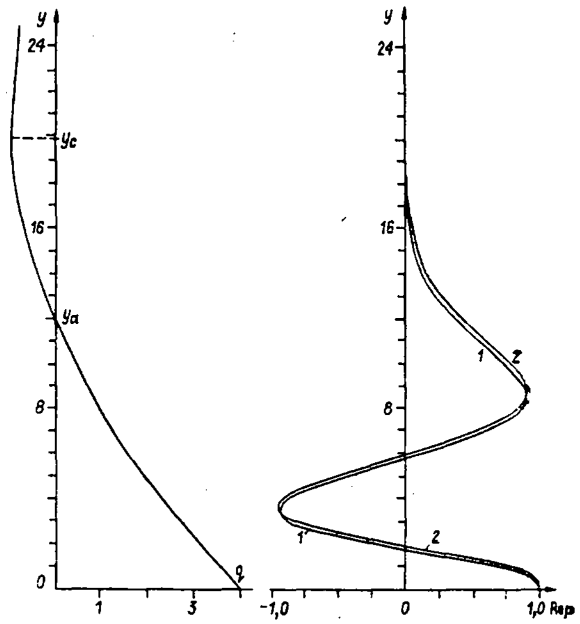

Fig. 3

$= (-q_a')^{1/6}\alpha^{-1/6}\pi^{-1/2}B_2,\ A_2 = (-q_a')^{1/6}\alpha^{-1/6}\pi^{-1/2}B_1/2$. Combining solutions (1.5), (1.8), and (1.6) between zones 2, c, and 3 leads to the relations

$$D_{1,2} = \alpha^{-1}T_c^{-1/2}U_c' \exp(\pm\alpha S)\, A_{1,2},\quad D_{1,2} = \alpha^{-1}T_c^{-1/2}U_c' E_{1,2}, S = \int_{y_a}^{y_c} \sqrt{-q}\,dy$$

$$\{T_c = T(y_c),\ U_c' = U'(y_c)\}.$$

Satisfying the boundary conditions, we obtain the dispersion relation

$$\cos\left[\alpha\int_0^{y_a}\sqrt{q}\,dy + \frac{\pi}{4}\right] = 0,\quad \alpha_m = \frac{\frac{\pi}{4} + \pi m}{\int_0^{y_a}\sqrt{\frac{(U-c)^2 M^2}{T} - \varkappa^2}\,dy},\quad m \to \infty. \tag{1.9}$$

The critical layer is absent for subsonic oscillations with the phase velocity $1 < c < 1 + \varkappa/M$. In this case, we can construct an expansion which is uniformly valid within the regions 1, a, and 2 [6]. We represent the solution in the form

$$u(y) = C_1(y)\, w(\alpha^{2/3}\xi(y)) + \alpha^{-1/3}C_2(y)\, w'(\alpha^{2/3}\xi(y)),$$

$$C_1 = \sum_{n=0}^{\infty}\alpha^{-n}A_n(y),\quad C_2 = \sum_{n=0}^{\infty}\alpha^{-n}B_n(y).$$

Here $\xi(y)$, A_n, B_n are unknown functions; $w(t)$ is the solution of the Airy equation $w'' - tw = 0$. In the principal approximation, we find the solution which decays at $y \to \infty$: $u(\xi) = \xi^{1/4}(-q)^{-1/4}C_0 Ai(\xi),\ (2/3)\,\xi^{3/2} = \alpha\int_{y_a}^{y}\sqrt{-q}\,dy$ (C_0 is a constant). Satisfying the boundary condition $u'(0) = 0$, we obtain dispersion relation (1.9).

2. The solutions constructed above indicate that shortwave free oscillations in a supersonic boundary layer have the following properties. Acoustic quasiwaves are formed in region 1 between the turning point and the wall. The quasiwaves are reflected from the boundaries with a coefficient equal to unity. The critical layer does not affect the dispersion relation in the principal approximation. Instead, it leads only to distortion of the eigenfunction p(y), changing the sign of the pre-exponential multiplier. It follows from Eq. (1.9) that, in the principal approximation, shortwave modes are neutral for the entire range of phase velocities examined. This is due to the fact that the critical layer, being a source

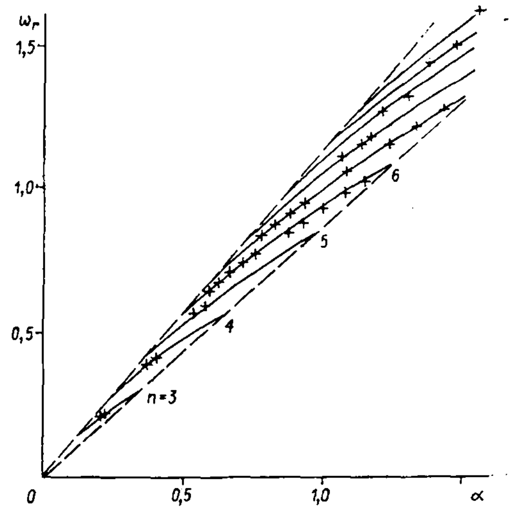

Fig. 4

of inviscid instability, weakly affects the free oscillations in accordance with an exponential law. An increase in m should be accompanied by a rapid decrease in the growth increments c_i, so that unstable modes will change into neutral oscillations of the acoustic type. This conclusion was confirmed by numerical calculations performed for a bounary layer on a thermally insulated plate in a gas flow with the adiabatic exponent $\gamma = 1.4$ and the Prandtl number 0.72. Figure 2 shows the time increments $\omega_i = \text{Im}(\omega)$ of two-dimensional perturbations ($\varkappa = 1$) in relation to the wave number α for the first four modes. The modes are numbered in accordance with [3]. The value of n corresponds to the number of the curve and is connected with m in (1.9) by the condition $n = m + 2$. The calculation was performed for $M = 8$ and a temperature on the external boundary of the boundary layer $T_\infty = 311$ K. It is evident that, beginning with $n = 3$, the increments decrease sharply. For the fourth mode, the maximum values $\omega_i = 1.9 \cdot 10^{-5}$ nearly correspond to a neutral perturbation. Thus, as regards shortwave disturbances, a supersonic boundary layer exhibits the properties of an acoustic waveguide. The boundaries of the waveguide are the layer near the acoustic point and the surface in the flow.

To check the validity of the asymptotic theory, we compared the free oscillations of the fourth mode with the results of a direct numerical calculation. Figure 3 shows distributions of the amplitude of pressure $\text{Re}(p(y))$ normalized with the condition $|p(0)| = 1$. The coordinate y is made dimensionless with respect to $L = (\nu_\infty x^*/U_\infty)^{1/2}$, where ν_∞ is the kinematic viscosity and x^* is the distance from the leading edge of the plate to the section in which the boundary layer is being examined. Curve 1 corresponds to the function calculated numerically for $c = 0.959$, $\alpha = 0.511$ and 2 and was obtained from the asymptotic solution for the same phase velocity. The function $q(y)$ is shown on the left, and the turning point and critical point are indicated.

Figure 4 shows the relations $\omega_r(\alpha)$ calculated from the dispersion relation (1.9) for $n = 3$-11 (solid lines). The x's show results of the numerical calculations, while the dashed lines show the limiting phase velocities $c = 1 \pm 1/M$. Beginning with $n \geq 3$, the asymptotic theory agrees satisfactorily with the direct numerical calculation.

It should be noted that the above asymptotic relations can be used as an initial approximation for calculations of eigenfunctions and eigenvalues of higher modes with allowance for viscosity. They also make it possible to perform simple approximate calculations of the characteristics of inviscid shortwave perturbations.

3. To study certain properties of the dependence of the eigenvalues on the wave number $\omega(\alpha)$, we analyze the group velocity $\partial\omega/\partial\alpha$. By making the substitution $p = \varphi_1$, $\varphi_1' = \varphi_2$, we reduce (1.1) to a system of second-order equations

$$\varphi_1' = \varphi_2, \quad \varphi_2' = -a\varphi_2 - b\varphi_1,$$
$$\varphi_1(\infty) = \varphi_2(0) = 0,$$

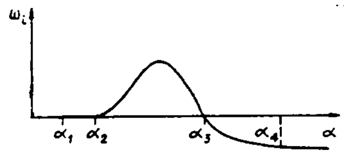

Fig. 5

$$a=\frac{T'}{T}-\frac{2U'}{U-c},$$
$$b=\alpha^2\left[\frac{M^2(U-c)^2}{T}-\varkappa^2\right]. \tag{3.1}$$

Determining the scalar product of two vector functions $(A, B)=\int\limits_0^\infty(A_1\bar{B}_1+A_2\bar{B}_2)\,dy$, we obtain the conjugate problem

$$\psi_1'=\bar{b}\psi_2,\ \psi_2'=-\psi_1+\bar{a}\psi_2,\ \psi_2(\infty)=\psi_1(0)=0. \tag{3.2}$$

The bar above the symbols in (3.2) denotes complex conjugation. Differentiating (3.1) with respect to the parameter α, we have

$$\varphi_{1\alpha}'-\varphi_{2\alpha}=0,\ \varphi_{2\alpha}'+a\varphi_{2\alpha}+b\varphi_{1\alpha}=-a_\alpha\varphi_2-b_\alpha\varphi_1,\ \varphi_{1\alpha}(\infty)=\varphi_{2\alpha}(0)=0$$
$$(\varphi_{1\alpha}=\partial\varphi_1/\partial\alpha,\ \varphi_{2\alpha}=\partial\varphi_2/\partial\alpha). \tag{3.3}$$

System (3.2) constitutes an inhomogeneous boundary-value problem with homogeneous boundary conditions. Its solution requires that the right side be orthogonal to the solution of the conjugate problem $\int\limits_0^\infty[a_\alpha\varphi_2+b_\alpha\varphi_1]\bar{\psi}_2dy=0$. From this, we obtain the following relation for the group velocity

$$\frac{\partial\omega}{\partial\alpha}=\frac{\int\limits_0^\infty\left[\frac{U'c}{\alpha(U-c)^2}\varphi_2\bar{\psi}_2+\left(\frac{M^2(U-c)U}{T}-\varkappa^2\right)\alpha\varphi_1\bar{\psi}_2\right]dy}{\int\limits_0^\infty\left[\frac{U'}{\alpha(U-c)^2}\varphi_2\bar{\psi}_2+\frac{M^2(U-c)}{T}\alpha\varphi_1\bar{\psi}_2\right]dy}. \tag{3.4}$$

We designate $\psi=\bar{\psi}_2$. Then system (3.2) can be written in the form

$$\psi''-a\psi'+(b-a')\psi=0,\ \psi(\infty)=0. \tag{3.5}$$

It is easily shown that if the direct eigenvalue problem is solved, then any solution of problem (3.5) will correspond to the boundary condition on the wall $\psi(0)=0$. Making the substitution $\psi(y)=T^{1/2}(U-c)^{-1}u(y)$, we find that the function u satisfies system (1.3). Thus, we have established the following connection between the pressure perturbation and the function ψ: $\psi(y)=p(y)T/(U-c)^2$. Using this relation, we have the asymptote of Eq. (3.4) with large αM:

$$\frac{\partial\omega}{\partial\alpha}=\frac{\int\limits_0^\infty\frac{Up^2}{U-c}dy}{\int\limits_0^\infty\frac{p^2}{U-c}dy}+O(\alpha M)^{-1},\quad \alpha M\to\infty. \tag{3.6}$$

Let us examine the behavior of $\omega(\alpha)$ with real wave numbers α. For a phase velocity $c_r>1$, the critical layer is absent and, as was shown in [8], $c_i=0$. In this case, Im $p(y)=0$, and it follows from (3.4) and (3.6) that $0<\partial\omega_r/\partial\alpha<1$. An increase in α is accompanied by an increase in the real part ω_r and a decrease in phase velocity, since $\partial c/\partial\alpha=(\partial\omega/\partial\alpha-c)/\alpha<0$. Thus, the perturbation is neutral between the points α_1 ($c=1+\varkappa/M$) and α_2 ($c=1$). At the point α_1, the eigenvalue of the discrete spectrum combines with the branch of the continuous spectrum determined by the condition $c>1+\varkappa/M$. The critical layer is present in the boundary layer for $\alpha>\alpha_2$ ($c_r<1$), so that instability does occur $\omega_i > 0$. It follows from the numerical calculations that at a certain value $\alpha_3 > \alpha_2$ $(1 - \varkappa/M < c(\alpha_3) < 1)$, a neutral regime $\omega_i = 0$ is again attained. In this case the expression for group velocity has the form

$$\frac{\partial\omega}{\partial\alpha} = \frac{\int_0^\infty \frac{Up^2}{U-c} dy \int_0^\infty \frac{p^2}{U-c} dy + \frac{\pi^2 p_c^4 c}{U_c'^2} - i\pi \frac{p_c^2}{U_c'} \int_0^\infty p^2 dy}{\left(\int_0^\infty \frac{p^2}{U-c} dy\right)^2 + \frac{\pi^2 p_c^4}{U_c'^2}}. \quad (3.7)$$

Here, all of the integrals are taken in the sense of the eigenvalue. The critical point y_c is circumvented below in the complex plane y. It follows from (3.7) that for $\partial\omega_r/\partial\alpha < 1$, $\partial\omega_i/\partial\alpha < 0$ at $\omega_i = 0$, $0 < c_r < 1$. Thus, the graph of $\omega_i(\alpha)$ at point α_3 has a negative slope.

At $\alpha > \alpha_4$ $(c_r(\alpha_4) = 1 + \varkappa/M)$, a second turning point appears near the upper boundary of the boundary layer, and the pressure pulsations in the external flow oscillate with respect to y. Figure 5 shows the quantitative relation $\omega_i(\alpha)$ and the characteristic points α_1, α_2, α_3, α_4. These results agree with the numerical calculation in Fig. 2. Using the definitions of the points α_1 and α_2 and dispersion relation (1.9), it is easy to calculate the boundaries of the neutral and unstable regions for each specific case.

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